AD NUMBER

AD216369

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AUTHORITY

ONR ltr, 13 Sep 1977

Division of Engineering
BROWN UNIVERSITY
PROVIDENCE, R. I.

AFCRC-TN-59-354
ASTIA Doc. No. AD-216369

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V. M. PAPADOPOULOS

Air Force Cambridge Research Center
Air Research and Development Command
Contract AF19(604)-4561
Scientific Report AF 4561/4
May 1959

AFCRC - TN - 59 - 354
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Contract Monitor Dr. Werner W. Gerbes

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"The research reported in this document has been sponsored in part by the Electronics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, and by the Office of Naval Research under contract Nonr 562(24). The publication of this report does not necessarily constitute approval by the Air Force of the findings or conclusions contained herein."

Contract title: Research Directed toward the study of Radiation of Electromagnetic Waves

Contract number: AF 19(604)-4561

[redacted]

ABSTRACT

A method for solving problems of plane pulse diffraction by a perfectly conducting wedge is described. The method is extended to give results when a conductive half-plane lies on the surface between two distinct isotropic media of different dielectric properties.

THE DIFFRACTION AND REFRACTION OF PLANE PULSES

INTRODUCTION

Both in the theory of acoustics which describes the propagation of infinitessimal disturbances, and in the closely related electromagnetic theory, research into the propagation of aperiodic disturbances has been overshadowed by research into the behavior of disturbances varying sinusoidally with time. It is possible to use a Fourier integral method to investigate aperiodic disturbances within a single homogeneous medium, but Craggs (1956, 1957) points out that such a method is useless in cases when total reflexion is to be expected at the surface of discontinuity between two homogeneous media.

Friedlander (1958) has recently published an excellent monograph in which he discusses the behavior of acoustic pulses. He does not, however, give much attention to the refraction of pulses in spite of having derived (Friedlander, 1948) an interesting result in this subject. We already know what combination of reflected and refracted pulses can travel along an infinite plane surface between two media with an incident pulse having a step-function time dependence. Friedlander found the nature of the disturbance in the second medium when the angle of incidence of the pulse in the first medium is large enough for total reflexion to occur. The manner in which such a steadily traveling disturbance can be set up is not known.

The first aim of this research is to learn how a plane pulse in one medium sets up a disturbance in a second even in the case of total reflexion. The second aim is to find a solution to the problem of diffraction in the presence of a refracting surface. There is no known solution for this problem in the case of steady sinusoidal excitation, even in the simplest extention of the classical problem of half-plane diffraction, when we take the obstacle to lie on the surface separating two distinct homogeneous regions. The powerful method of Wiener and Hopf gives no usable result.

Craggs (1956, 1957) has shown that an assumption of dynamic similarity may be useful in this type of problem. Thus when investigating a scalar function $s(r,\theta,t)$ which satisfies within a single medium the wave equation

$$c^2 \nabla^2 s = \partial^2 s / \partial t^2 , \qquad (1)$$

where c is the constant velocity of propagation, we may take $\lambda = r/t$ to be a new independent variable. Then equation 1 becomes

$$\lambda^2 \left(1 - \frac{\lambda^2}{c^2}\right) \frac{\partial^2 s}{\partial \lambda^2} + \lambda \left(1 - \frac{2\lambda^2}{c^2}\right) \frac{\partial s}{\partial \lambda} + \frac{\partial^2 s}{\partial \theta^2} = 0 . \qquad (2)$$

For $\lambda > c$, equation 2 is hyperbolic; it may be reduced to the canonical form

$$\frac{\partial^2 s}{\partial \mu^2} - \frac{\partial^2 s}{\partial \theta^2} = 0 ,$$

by the transformation $\lambda = c \sec \mu$. The general solution of this equation is

$$s = f(\mu - \theta) + g(\mu + \theta) , \qquad (3)$$

where f and g are arbitrary functions, constant on the family of characteristics $\mu - \theta$ = constant, and $\mu + \theta$ = constant, respectively. In the plane with polar co-ordinates (λ, θ) these characteristics are tangents to the circle $\lambda = C$; it is necessary to regard the parts of a tangent on opposite sides of the point of contact as distinct and of opposite family.

For $\lambda < c$, equation 2 is elliptic. It reduces to

$$\frac{\partial^2 s}{\partial \nu^2} + \frac{\partial^2 s}{\partial \theta^2} = 0 , \qquad (4)$$

under the transformation $\lambda = c \operatorname{sech} (-\nu)$. The sign of ν is chosen so that $\nu \to \ln (\lambda/c)$ as $\lambda \to 0$. It is convenient to introduce the conjugate harmonic function $\sigma(\nu, \theta)$ so that the function $w = s + i\sigma$ is analytic in the region of the (ν, θ) plane which corresponds to $\lambda < c$. It follows that $\partial s/\partial \theta = -\partial \sigma/\partial \nu$. and $\partial s/\partial \nu = \partial \sigma/\partial \theta$.

Equations 3 and 4 show the technical start in the solution of a number of physical problems. In electromagnetic theory, by considering problems in cylindrical polar co-ordinates (r, θ, z) in which the field components are independent of z, $s(r, \theta, t)$ may represent either the field component E_z or B_z. From Maxwell's equations we can write down equations for the remaining field components in both an E- and an H-polarization. These are

$$\lambda^2 \frac{\partial B_r}{\partial \lambda} = \frac{\partial E_z}{\partial \theta}, \quad -\lambda \frac{\partial B_\theta}{\partial \lambda} = \frac{\partial E_z}{\partial \lambda}, \quad (\text{E-mode}), \qquad (5)$$

and

$$\lambda^2 \frac{\partial E_r}{\partial \lambda} = -c^2 \frac{\partial B_z}{\partial \theta}, \quad \lambda \frac{\partial E_\theta}{\partial \lambda} = c^2 \frac{\partial B_z}{\partial \lambda}, \quad (\text{H-mode}) \qquad (6)$$

The problem to be examined all involve infinite prisms with the line $r=0$ for apex. We set up a system of plane pulses traveling towards the apex so that behind the pulse fronts all the appropriate boundary conditions are satisfied. We take the moment when the incident pulses reach the apex to be $t = 0$, and we assume, in the absence of a fundamental length in the geometry, that the subsequent disturbance is one in which there is dynamic similarity. We take the prism to be bounded externally by perfectly conducting walls.

First we shall derive in Part 1 the results for pulse diffraction within a homogeneous prism; these results may be compared with those established previously (see e. g. Friedlander, 1958).

We shall then extend the method in Part 2 to find a solution in the case when a half-plane with its edge at the origin lies on the surface which separates two distinct isotropic media. We set as the restriction on the angle of incidence the condition that the incident pulses shall be contained in one medium only.

PART I.

PULSE DIFFRACTION IN A HOMOGENEOUS MEDIUM
BY A HALF-PLANE AND BY A WEDGE

Consider the case of a perfectly conducting half-plane in the diffraction problem where $s = B_z$ and the appropriate boundary condition on the half-plane is that $\partial s/\partial\theta = 0$. We set up a plane pulse traveling with velocity c, and with a step function time dependence of unit amplitude. There are two cases which need separate examination, that (case a) with the pulse traveling towards the edge of the half-plane from the unbifurcated region at an arbitrary angle, and that (case b) with the pulse traveling along the half-plane towards the edge, at oblique incidence. We take the moment when the incident pulse meets the edge to be t=0. The distinction between the two cases is needed since, in order to satisfy the boundary condition on the half-plane for $t < 0$ in case (b), we must introduce the associated reflected pulse. The conditions on our problem at t=0 are shown in the figures 1. In each case we take the incident pulse to make an angle α with the normal to the half-plane with $\alpha = \pi/2$. We fix the co-ordinates by taking the origin at the edge of the half-plane $\theta = 0$ or $\theta = 2\pi$.

Under the assumption of dynamic similarity, we may examine the nature of the solution for $\lambda > c$ in the (λ, θ) plane. Since for $t \to 0_+$, $\lambda \to \infty$ we find that the initial conditions of the problem fix the boundary conditions on equation 3 for $\lambda \to \infty$. The use of characteristics theory shows us that for $\lambda >$ c there are a number of uniform regions separated by lines of discontinuity which are depicted in figure 2. The results are precisely those of geometrical optics. The lines of discontinuity are pulse fronts. These results determine the value of s on $\lambda =$ c.

Thus on $\lambda = c$,

$$s = 2 \quad \text{for} \quad 0 < \theta < \beta ,$$

$$s = 1 \quad \text{for} \quad \beta < \theta < \gamma ,$$

$$s = 0 \quad \text{for} \quad \gamma < \theta < 2\pi ,$$

where $\beta = \alpha$ in case (a), $\beta = \pi - \alpha$ in case (b), and $\gamma = 2\pi - \beta$.

For the region $\lambda < c$ it follows from equation 4 that the problem becomes that of finding a function $w(\nu + i\theta) = s + it$ which satisfies the boundary condition $\partial s/\partial\theta = 0$ on $\theta = 0, \theta = 2\pi$, and on $\nu = 0$ (i.e. $\lambda = c$), and which is analytic within the semi-infinite strip $\nu > 0$, $0 < \theta < 2\pi$. We make the conformal transformation $f = \operatorname{sech} \nu + i\theta/2$, which maps this semi-infinite strip into the upper half of the complex f-plane. Then the function $\partial w/\partial f$, regarded as a function of the complex variable $f = \xi + i\eta$ must satisfy the following conditions:

i) To correspond to the boundary condition on the half-plane, it follows that $\partial w/\partial f$ is real on the segment of the real axis $|\xi| < 1$.

ii) To correspond to the boundary condition that s is piecewise constant on the circle $\lambda = c$, it follows that $\partial w/\partial f$ is imaginary on the remainder of the real axis ($|\xi| > 1$), except perhaps at points where s is discontinuous.

iii) Singularities of $\partial w/\partial f$ are to be expected only at the points $f = \pm 1$, $f = 0$, as well as at the points $f = \sec \alpha/2$, $\sec (2\pi - \alpha)/2$ in case (a), and at $\sec (\pi \pm \alpha)/2$ in case (b).

iv) Since on physical grounds the magnetic field in this polarization is assumed finite, no singularity of $\partial w/\partial f$ may be of higher order than a simple pole.

v) The point at infinity in the f-plane corresponding to the ordinary point $\lambda = c$, $\theta = \pi$ in the (λ, θ) plane must be an ordinary point for w and $\partial w/\partial f$, so that $w = 0\ (1/f^{1+\delta})$, with $\delta > 0$ as $f \to \infty$.

The most general function which satisfies these conditions is

$$\pi \frac{\partial \omega}{\partial f} = \frac{i}{(f^2-1)^{1/2}} \left[\frac{B'}{f - \sec \beta/2} + \frac{C'}{f - \sec \gamma/2} \right], \qquad (7)$$

where B' and C' are real constants which are determined by the discontinuities in s at the points B and C in the (λ, θ) plane. To find the constant B', we integrate $\partial\omega/\partial f$ round the small semicircle f = sec $\beta/2 + \exp i\phi$, $0 \leq \phi \leq \pi$, by the method of residues; we find C' in a similar manner. It follows that B' $\cot \beta/2$, C' $\cot \gamma/2$ must equal the discontinuity in s on the circle at the points B and C respectively taken in the direction of increasing θ.

Hence,

$$\pi \frac{\partial \omega}{\partial f} = \frac{-i}{(f^2-1)^{1/2}} \left[\frac{\tan \beta/2}{f - \sec \beta/2} + \frac{\tan \gamma/2}{f - \sec \gamma/2} \right], \qquad (8)$$

The integration of this equation is straightforward. We find, after putting $\gamma = 2\pi - \beta$ that

$$\pi\left[\omega(r,\theta) - \omega(0,\theta)\right] = -i \ln \left\{ \frac{\tan\left(\frac{\beta+\theta-ir}{4}\right) \tan\left(\frac{\beta-\theta}{4}\right)}{\tan\left(\frac{\beta+\theta}{4}\right) \tan\left(\frac{\beta-\theta+ir}{4}\right)} \right\}.$$

The real part of this equation is

$$\pi\left[s(r,\theta) - s(0,\theta)\right] = -\tan^{-1}\left[\sinh \frac{r}{2} \operatorname{cosec} \frac{\beta-\theta}{2}\right] - \tan^{-1}\left[\sinh \frac{r}{2} \operatorname{cosec} \frac{\beta+\theta}{2}\right]. \qquad (9)$$

This formula for the diffracted field is exactly that derived and described by Friedlander by a Green's function method in his Chapter 5.

The results for the E-polarization may be found just as easily. For the hyperbolic region in the (λ, θ) plane it is clear that the reflected pulse must annul the incident field instead of reinforcing it. Hence with reference to figure 2 the region with s=2 becomes a region with s = 0, and the discontin-

uity in s at B is changed accordingly. For the elliptic region, we need only change the condition 1. On the half-plane s must vanish, so that its tangential derivate is zero. It follows that on the segment of the real ξ -axis $|\xi| < 1$, $\partial\omega/\partial\xi$ is imaginary.

The function satisfying the new set of conditions and with the correct discontinuities is

$$\pi\frac{\partial\omega}{\partial\xi} = \frac{i}{\xi}\left[\frac{\sec\beta/2}{\xi - \sec\beta/2} - \frac{\sec\gamma/2}{\xi - \sec\gamma/2}\right]. \qquad (10)$$

Hence, the final expression for s is

$$\pi\left[s(v,\theta) - s(0,\theta)\right] = \tan^{-1}\left[\sinh\frac{v}{2}\operatorname{cosec}\frac{\beta-\theta}{2}\right] - \tan^{-1}\left[\sinh\frac{v}{2}\operatorname{cosec}\frac{\beta+\theta}{2}\right]. \qquad (11)$$

The analysis of the problem of pulse diffraction by a perfectly reflecting wedge is carried out in the same way. We exclude, from this brief discussion, any case in which the incident front touches both walls of the wedge, in order to avoid the complication of multiple reflexion before the apex is reached. This leaves us with the general problem for which the wedge angle $\phi\pi$ is greater than $\pi/2$, and the angle $\alpha\pi$ which an incident pulse makes with the normal to one face of the wedge is smaller than $\pi(\phi - \frac{1}{2})$.

In such a situation we depict an initial state in figure 3. We use the method of characteristics to determine the solution in the hyperbolic region. This is depicted in figure 4, hence we note that on the optic circle $\lambda = c$.

$$s = 4 \quad \text{for} \quad 0 < \theta < \pi(1-\alpha-\phi),$$

$$s = 3 \quad \text{for} \quad \pi(1-\alpha-\phi) < \theta < \pi(1+\alpha-\phi),$$

and $s = 2 \quad \text{for} \quad \pi(1+\alpha-\phi) < \theta < \pi\phi.$

This particular situation is valid if $\phi + \alpha < 1$.

To find the disturbance in the elliptic region, we use the conformal transformation ζ = sech $(\nu + i\,\theta)/\phi$ which maps the sector of the circle $\lambda < c$ $0 < \theta < \phi\pi$ into the upper half ζ -plane. The conditions which must be satisfied by the function $\partial\omega/\partial\zeta$ are exactly those in the half-plane diffraction problem, and indeed, equation 10 gives the formula for $\partial\omega/\partial\zeta$ in the wedge problem if we take $\beta = 2(1-\alpha-\phi)/\phi$ and $\gamma = 2(1+\alpha-\phi)/\phi$. Integration of equation 10 in this case also leads us to Friedlander's equation 5.5.7. (Friedlander, 1958).

It remains to be emphasized that this technique and the conical flow method of Buseman (1942) are very closely related. Keller and Blank (1952) described the results of this method very fully for all possible situations in the problem of pulse diffraction by a wedge.

PART II.

PULSE DIFFRACTION AND REFRACTION IN A COMPOSITE SYSTEM

INTRODUCTION

The value of the technique described in part I does not lie in the fact that it provides yet another, slightly different means of solving the classical half-plane problem. Indeed, the only reason for the inclusion of the details in part I is to confirm the value of the assumption of dynamic similarity in a diffraction problem.

We have already shown that in these two dimensional problems the problem of finding the vector solutions of Maxwells' equations reduces to that of finding solutions to the scalar wave equation with the boundary coordinates $s = 0$ or $\partial s/\partial\theta = 0$. This situation is exactly analogous to that in acoustics if we take s to represent the condensation, so that the fluid velocity $\underline{q}$ is given by

$$\partial \underline{q} / \partial t = - c^2 \underline{\nabla} s \qquad (12)$$

and hence the radial and transverse components of velocity (u,v) are given by

$$\lambda \frac{\partial u}{\partial \lambda} = c^2 \frac{\partial s}{\partial \lambda} , \quad \lambda^2 \frac{\partial v}{\partial \lambda} = c^2 \frac{\partial s}{\partial \theta} . \qquad (13)$$

In this section the method is extended to give the solution in the case when a plane pulse is diffracted by a half-plane which lies between two regions of different physical properties. The half-plane $\theta = \pm\pi$ is either perfectly reflecting or perfectly absorbent. The velocity of sound in the fluid of density ρ is c, and wherever qualification is necessary, we show which medium we are referring to by the suffixes 1 or 2.

There are several aspects of the physical problem which are of particular interest. The singularity at the apex of the half-plane is the only known result

in the steady state problem. Meixner (1954), who examined the power series expansion of the solution of the reduced wave equation corresponding to a sinusoidal time dependence found the edge singularity in this geometry to be unaffected by the discontinuity in physical properties.

The manner in which a disturbance passes from one medium into another has not been discussed before. Of course, the Fresnel coefficients for steady plane pulse reflexion and refraction at a plane surface are well known, but these coefficients are without meaning when total reflexion occurs. Friedlander (1948) found the form of the disturbance in this case, but since both these situations are essentially steady ones, they give no insight into the transient problem.

Craggs (1956, 1957), who has provided the greater part of the basis for the method of solution in this paper, examined a situation involving two right-angled wedges. No diffraction occurs in this case, and there is no singularity at the common apex; it appears that his method need modifying when diffraction effects occur.

SECTION 1

A HALF-PLANE ON THE SURFACE BETWEEN TWO MEDIA

We shall first examine in detail the case when the diffracting obstacle is a perfect reflecting half-plane ($r > 0$, $\theta = \pm \pi$) lying on the plane of separation between two media. We consider the situation in which the velocity of sound c_1, in medium 1 ($r > 0$, $0 > \theta > -\pi$) is greater than that (c_2) in medium 2 ($r > 0$, $0 < \theta < \pi$). We put $c_2 = m c_1$, and $\rho_2 = k \rho_1$.

The problem is to examine the propagation of a plane pulse in medium 2 which makes an angle ϕ with the normal to the half-plane. As in part I, in order to make the assumption of dynamic similarity plausible, we must set up as an initial state one which satisfies all the appropriate boundary conditions. These conditions are that $\partial s_1/\partial\theta = 0$ for $\theta = -\pi$ and $\partial s_2/\partial\theta = 0$ for $\theta = \pi$. The conditions at the common surface between the two media are found from the continuity of pressure and normal velocity. Thus for $r > 0$, $\theta = 0$, $s_1 = m^2 k s_2$ and $\partial s_1/\partial\theta = m^2 \partial s_2/\partial\theta$. In order to avoid having to set up a refracted disturbance before the pulse reaches the edge of the half-plane, we must restrict consideration to the case when the pulse arrives from the second quadrant of the (r, θ) plane; this pulse is accompanied by the reflected pulse. The initial state is depicted in figure 5. The assumption of dynamic similarity having been made, we must now find suitable solutions for equations 3 and 4.

SECTION 2

THE SOLUTION IN THE HYPERBOLIC REGION

The solution in the greater part of the hyperbolic region is determined by the values of s as $\lambda \rightarrow \infty$, that is by the initial conditions of the problem, and it is found by the method of characteristics. There are two distinct situations.

according as $\phi <$ or $> \psi$, where $\psi = \sec^{-1}(1/m)$ is the critical angle; for $\phi < \psi$ total reflexion occurs. In the (λ, θ) plane the hyperbolic region lies outside the sonic circle $\lambda = c_1$ in medium 1 and $\lambda = c_2$ in medium 2, and the values of s in this region are shown in figures 6 and 7. The initial fronts are represented by the lines DI and ER. DE is the front of the pulse reflected at the refracting surface, and when the point D is outside the larger sonic circle, DT is the front of the refracted pulse. The line BC which is a characteristic line, is the front of the disturbance which having entered medium 1 moves along the interface at grazing incidence and is refracted back into medium 2. The solution at this stage is fully determined in the whole region outside the sonic circles except within the triangle ABC. Inside this region we introduce the unknown function $g(\mu_2 + \theta)$ to represent the part of s affected by the disturbance in the elliptic region of medium 1. (This is the function g of equation 3, with the function f a constant). The constants R and T in the normally refracting case (figure 7) are the Fresnel coefficients given by

$$R = \frac{\tan\phi - k(m^2\sec^2\phi - 1)^{1/2}}{\tan\phi + k(m^2\sec^2\phi - 1)^{1/2}}, \qquad (14)$$

$$T = \frac{2m^2 k \tan\phi}{\tan\phi + k(m^2\sec^2\phi - 1)^{1/2}}.$$

From figure 6 we see that on $\lambda = c_2$ in medium 2

$$s_2 = 1 + g \quad \text{for} \quad 0 < \theta < \phi,$$
$$s_2 = 2 + g \quad \text{for} \quad \phi < \theta < \psi,$$
$$s_2 = 2 \quad \text{for} \quad \psi < \theta < \pi,$$

while $s = 0$ on $\lambda = c_1$ in medium 1. From figure 7 we see that on $\lambda = c_2$ in medium 2.

$$s_2 = R + g \quad \text{for} \quad 0 < \theta < \psi,$$
$$s_2 = 1 + R \quad \text{for} \quad \psi < \theta < \phi,$$
$$s_2 = 2 \quad \text{for} \quad \phi < \theta < \pi,$$

while on $\lambda = c_1$ in medium 1

$$S_1 = T \quad \text{for} \quad 0 > \theta > -\chi$$

$$S_1 = 0 \quad \text{for} \quad -\chi > \theta > -\pi$$

where $\chi = \sec^{-1} m \sec \phi$.

SECTION 3

THE ELLIPTIC REGION OF THE (λ, θ) PLANE

We use the conformal transformation

$$\zeta_1 = \xi_1 + i\eta_1 = \operatorname{sech}(\nu_1 + i\theta)$$

to map the inside of the semicircle $\lambda = c_1, -\pi < \theta < 0$ into the lower half of the complex ζ_1-plane, and the transformation

$$\zeta_2 = \xi_2 + i\eta_2 = \operatorname{sech}(\nu_2 + i\theta)$$

to map the inside of the semicircle $\lambda = c_2$, $0 < \theta < \pi$ into the upper half of the complex ζ_2-plane. The points F. O. A. E. and C in medium 2 are mapped into the points

$\zeta_2 =$ -1, 0, 1, sec ϕ, and 1/m respectively while the points G, O, A, D, (or T), and B in medium 1 are mapped into the points $\zeta_1 =$ -1, 0, m, m sec ϕ and 1.

Consider first the solution in the lower half of the ζ_1-plane. Since s is piecewise constant on the semicircle $\lambda = c_1$ it follows as in section I that

i) $\partial w_1/\partial\zeta_1$ is imaginary on the section of the real axis $|\xi_1| > 1$.

From the condition that $\partial s/\partial\theta = 0$ on O G, it follows that

ii) $\partial w_1/\partial\zeta_1$ is real on the segment of the real axis $0 < \xi_1 < 1$.

On the upper side of the line AB we know that the solution takes the form $S = g(\mu_2 + \theta)$ + constant. It follows from this solution and the condition of continuity of pressure and normal velocity that

$$\frac{\partial s_1}{\partial\lambda} = m^2 k \frac{\partial s_2}{\partial\lambda} = m^2 k \frac{\partial\mu_2}{\partial\lambda}\frac{\partial s_2}{\partial\theta} = k\frac{\partial\mu_2}{\partial\lambda}\frac{\partial s_1}{\partial\theta} = -k\frac{\partial\mu_2}{\partial\lambda}\frac{\partial\tau_1}{\partial\nu_1} = -k\frac{\partial\mu_2}{\partial\nu_2}\frac{\partial\tau_1}{\partial\lambda}$$

Hence, we have the condition that $\dfrac{\partial s_1}{\partial\lambda} + k\dfrac{\partial\mu_2}{\partial\nu_1}\dfrac{\partial\tau_1}{\partial\lambda}$, i.e. that

$$\text{iii)} \quad Re\left[\frac{\partial w_1}{\partial\lambda}\left(1 - ik\frac{\partial\mu_2}{\partial\nu_1}\right)\right] = 0$$

Thus if $F(\zeta_1)$ is a real valued function on the segment $m < \zeta_1 < 1$, then on this segment

$$\frac{\partial \omega_1}{\partial \zeta_1} = F(\zeta_1) \Big/ 1 + \frac{i}{mk}\left(\frac{\zeta_1^2 - m^2}{1-\zeta_1^2}\right)^{1/2} \qquad (15)$$

The other conditions to be imposed, as in section 1, are that

iv) the only singularities of $\partial\omega_1/\partial\zeta_1$ are at points $\zeta_1 = \pm 1$, $+ m$, 0, as well as at the points $\zeta_1 = m \sec \phi$,

v) no singularity of $\partial\omega_1/\partial\zeta_1$ may be of higher order than a simple pole,

and

vi) the point at infinity is an ordinary point.

It is of interest to examine the function in the denominator in equation 15. This function is regular everywhere in the complex ζ_1-plane except on the real axis, it has branch points at $\zeta_1 = \pm m$, ± 1, and the branches of the radicals are chosen so that it has no zeros. The singularity at $\zeta_1 = -m$ is not one of those allowed under conditions iv, so that we must factorize this function in a manner which enables us to remove this singularity from the solution without changing its complex form on the real axis, for $m < \zeta_1 < 1$.

SECTION 4

THE FACTORIZATION OF THE COMPLEX FUNCTION

Consider the complex function $M(\zeta) = mk + i\left(\frac{\zeta^2-m^2}{1-\zeta^2}\right)^{1/2}\Big/ 1+mk$ which is the analytic continuation of the function defined on the segment of the real axis $m < \zeta < 1$. The properties of $L(\zeta) = \ln M(\zeta)$ are that

a) as $|\zeta| \to \infty$, $L(\zeta) = O(\zeta^{-2})$,

b) L (ζ) is regular in the infinite strip $-m < Rl(\zeta) < m$,

c) L (ζ) is a real function of ζ within the same strip, since the continuation of L (ζ) is $\ln\left\{\left[mk + \left(\frac{m^2-\zeta^2}{1-\zeta^2}\right)^{1/2} \Big/ 1 + mk\right]\right\}$, and

d) L (ζ) is single valued in the cut plane with one cut joining the points $-m, -1$ and another the points $+m, +1$.

If we take a rectangular contour C as shown in figure 9, with sides joining the points $\sigma_1 + iN$, $\sigma_2 + iN$, $\sigma_1 - iM$ and $\sigma_2 - iM$, then for $-m < \sigma_1 \leq Rl(z) \leq \sigma_2 < m$, we are able to use Cauchy's Integral to state that

$$L(\zeta) = \frac{1}{2\pi i}\int_C \frac{L(z)}{z-\zeta}\, dz$$

Here the integrand is sufficiently small as $|z| \to \infty$ to allow the limiting process, $N \to \infty$, $M \to \infty$ to be carried out separately at the upper and the lower end of the rectangle. The contribution from the ends of the rectangle is vanishingly small, so we may replace the contour integral by the sum of two line integrals

$$L(\zeta) = \frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \frac{L(z)}{z-\zeta}\, dz - \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{L(z)}{z-\zeta}\, dz,$$

$$= L_-(\zeta) - L_+(\zeta).$$

Each line integral $L_-(\zeta)$ and $L_+(\zeta)$ is uniformaly convergent for values of ζ taken within the infinite rectangle $|Rl(\zeta)| < m$. Taken separately each integral may be continued analytically into a complete half-plane since $L_-(\zeta)$ is regular for $Rl(\zeta) < m$ and $L_+(\zeta)$ is regular for $Rl(\zeta) > -m$.

Now since $L_+(\zeta) = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 - i\infty} \frac{L(z)}{z-\zeta}\, dz = \frac{1}{2\pi}\int_0^\infty \left[\frac{L(\sigma_1 + i\eta)}{\sigma_1 + i\eta - \zeta} + \frac{L(\sigma_1 - i\eta)}{\sigma_1 - i\eta - \zeta}\right] d\eta$

and since in $|z| < m$, $L(z)$ is a real function of z, for real values of ζ the integrand is the sum of complex conjugates, and the integral is therefore real for any real value of $\zeta > -m$. Likewise $L_-(\zeta)$ is real on the real axis for $\zeta < m$.

We may therefore write

$$\exp -L_-(\zeta) = \exp -L_+(\zeta) / M(\zeta) \qquad (16)$$

and we see that the function on either side of this equation is

1) regular in the half-plane $Rl(\zeta) < m$,

2) real on the real axis in $Rl(\zeta) < m$,

and 3) has the branch points and the complex behavior of $[M(\zeta)]^{-1}$ in the half-plane $Rl(\zeta) \geqslant m$.

It is also easy to show that $L_+(-\zeta) = -L_-(\zeta)$; this is a consequence of the even nature of the function $L(z)$.

In its present form, the integral $L_+(\zeta)$ is not very suitable for computation. However, we may add to this line integral one for which the integral is taken round a semicircle of infinite radius. The path of integration is thus closed without changing the value of the integral. The integrand has branch points at $z = -m$ and $z = -1$, within the contour. The path of integration may now be deformed into a loop encircling the cut, so we find that

$$\pi L_+(\zeta) = \int_m^1 \tan^{-1}\left[(z^2-m^2)^{1/2} / mk(1-z^2)^{1/2}\right] dz/z+\zeta \qquad (17)$$

This integrand may be expanded in an ascending series of powers of $(m+\zeta)^{-1}$; the integration of the coefficients of this expansion in ascending powers of $(1-m)$ is then easily carried out.

SECTION 5

THE SOLUTION IN THE ELLIPTIC REGION

The expression which represents the solution to the problem and which satisfies the conditions in section 3 may now be written down. The most general is that

$$\pi \frac{\partial \omega_1}{\partial \zeta_1} = \frac{\exp - L_+(\zeta_1)}{mk(1-\zeta_1^2)^{1/2} + i(\zeta_1^2 - m^2)^{1/2}} \left\{ \frac{A + B\left(\frac{\zeta_1 - m}{\zeta_1}\right)^{1/2}}{(\zeta_1 - m \sec \phi)(\zeta_1 - 1)} \right\}, \quad \text{-- (18)}$$

where the function $L_+(\zeta_1)$ has been defined in section 4.

The problem which remains is to find the disturbance in the elliptic region $\lambda < c_2,\ 0 < \theta < \pi$. Now equation 18 has been derived for general values of ζ_1 in the lower half-plane by means of the principle of analytic continuation, and we may in turn continue this solution into the upper half ζ_2-plane, because the solutions in these two regions are linked across the line OA, on which $\zeta_1 = m\zeta_2$.

Across this line OA the continuity conditions may be written in the form

$$m^2 k \frac{\partial s_2}{\partial \zeta_2} = \frac{\partial s_1}{\partial \zeta_2} \quad ; \quad \frac{\partial \tau_1}{\partial \zeta_2} = m^2 \left(\frac{1 - \zeta_2^2}{1 - m^2 \zeta_2^2} \right)^{1/2} \frac{\partial \tau_2}{\partial \zeta_2} . \qquad (19)$$

Since on the corresponding section of the real axis $0 < \zeta_2 < m$

$$\pi \frac{\partial \omega_1}{\partial \zeta_1} = \frac{\exp - L_+(\zeta_1)}{mk(1-\zeta_1^2)^{1/2} + (m^2 - \zeta_1^2)^{1/2}} \left\{ \frac{A - iB(m - \zeta_1/\zeta_1)^{1/2}}{(\zeta_1 - m \sec \phi)(\zeta_1 - 1)} \right\}, \qquad (20)$$

it follows from equations 19 and 20 that

$$m^2 k \pi \frac{\partial \omega_2}{\partial \zeta_2} = \frac{\exp - L_+(m\zeta_2)}{mk(1-m^2\zeta_2^2)^{1/2} + m(1-\zeta_2^2)^{1/2}} \left\{ \frac{A - ikB\left(\frac{1 - m^2\zeta_2^2}{\zeta_2(1+\zeta_2)}\right)^{1/2}}{(\zeta_2 - \sec \phi)(m\zeta_2 - 1)} \right\} \qquad (21)$$

on the section of the real axis $0 < \zeta_2 < 1$

The function $\left\{ 1 + \frac{1}{k} \left(\frac{1-\zeta_2^2}{1-m^2\zeta_2^2} \right)^{1/2} \right\} \exp + L_+(m\zeta_2)$ is regular in the half-plane Rl $(\zeta_2) < 1$, it is real on the real axis in this domain of regularity and it has the complex behavior and the branch points of the function $1 + \frac{i}{k} \left(\frac{\zeta_2^2 - 1}{1 - m^2 \zeta_2^2} \right)^{1/2}$ in the half-plane Rl $(\zeta_2) \geqslant -1$.

As in the case of the other elliptic region we expect the function $\partial w_2/\partial \zeta_2$ to have the properties that

1) it has singularities at $\zeta_2 = 0, \pm 1, + 1/m$ and $+ \sec \phi$,

2) the point at infinity is an ordinary point,

3) it is imaginary on the real ζ_2 - axis when $\zeta_2 > 1/m, \zeta_2 < -1$ and it is real on the real ζ_2 -axis when $-1 < \zeta_2 < 0$.

There is, however, a branch point at $\zeta_2 = -1/m$ in the expression on the right-hand side of equation 21. The only possible way in which the expression 18, which has the correct behavior in the lower half-plane, may be continued into an expression correct in the upper region is to take $A = 0$; there is, therefore, only the constant B which remains to be found.

SECTION 6

THE DETERMINATION OF THE CONSTANT B

It will be remembered that in the figures 4 and 5, we have distinct representations of the problem for the cases when $m \sec \phi < 1$ and when $m \sec \phi > 1$. In the case of supercritical incidence shown in figure 5 we may use the method of residues to relate the discontinuity on the sonic circle either at the point T or at the point E; we find the same result in each case. It follows that

$$T = \frac{2m^2 k \tan \phi}{\tan \phi + k(m^2 \sec^2 \phi - 1)^{1/2}} = \frac{B \exp - L_+(m \sec \phi) \, (\sec \phi - 1/\sec \phi)^{1/2}}{m \left[\tan \phi + k(m^2 \sec^2 \phi - 1)^{1/2} \right] (m \sec \phi - 1)}, \qquad (21)$$

and hence that

$$B \sin \phi/2 = 2^{1/2} m^3 k \exp + L_+ (m \sec \phi) \tan \phi (m \sec \phi - 1) \tag{22}$$

In the subcritical case with $m \sec\phi < 1$, the discontinuities at the points D and E in figure 6 are related because DE is a characteristic line in the hyperbolic region. At the point D the discontinuity in s_2 as λ increases is found by the continuity condition to be $\Delta s_2 = m^2 k(\Delta g - 1)$ where Δg is the jump in the function g across the line DE. The discontinuity Δs_1 at the point E is given, for increasing θ, by the equation $\Delta s_1 = 1 + \Delta g$

so that

$$\Delta s_2 = m^2 k (\Delta s_1 - 2) \tag{23}$$

The real part of the residue when we integrate the functions $\partial \omega_1/\partial f_1$ and $\partial \omega_2/\partial f_2$ round small semicircles with centres at the points $f_1 = \sec \phi$ and $f_2 = m \sec \phi$ determines the quantities Δs_1, and Δs_2. We find that

$$\Delta \omega_2 = \frac{iB \exp - L_+ (m \sec \phi) \left[(\sec \phi - 1) \sec \phi \right]^{1/2}}{\left[mk (1 - m^2 \sec^2 \phi)^{1/2} + i m \tan \phi \right] (m \sec \phi - 1)}, \tag{24}$$

and that

$$m^2 \Delta \omega_1 = - \Delta \omega_2 (1 - m^2 \sec^2 \phi)^{1/2} / \tan \phi \tag{25}$$

From these equations it follows that

$$m^2 \Delta s_1 \tan \phi + \Delta s_2 (1 - m^2 \sec^2 \phi)^{1/2} = 0 \tag{26}$$

so that

$$\Delta s_2 = \frac{2m^2 k \tan\phi}{\tan\phi + k\,(1 - m^2\sec^2\phi)^{1/2}} , \tag{27}$$

$$\Delta s_1 = \frac{2k\,(1 - m^2\sec^2\phi)^{1/2}}{\tan\phi + k\,(1 - m^2\sec^2\phi)^{1/2}} , \tag{28}$$

and

$$B \sin\phi/2 = 2^{1/2} m^3 k \exp L_+(m\sec\phi)\, \tan\phi\,(m\sec\phi - 1) . \tag{29}$$

A comparison between equations 26 and 27 and the corresponding discontinuities T and 1-R in the supercritical case, which are defined by equation 14, is instructive. It shows that modified Fresnel coefficients R^* and T^* given by the equations $T^* = \Delta s_2$ and $R^* = 1 - \Delta s_1$, may be used in the subcritical case to determine the discontinuities within a disturbance at the front DE of the reflected pulse.

Equations 18 and 21, with $A = 0$ and with B determined by the equations 22 and 29 give explicit formula for the derivatives of the functions w_1 and w_2. From these formulae it is easy to write down the expressions for both the spatial and the time derivatives of the condensation s in the elliptic regions; the values of s or the values of the velocity components may then be found by numerical integration.

One result which is worth writing down here is that the jump in s across the characteristic BC is

$$\exp\left\{\frac{L_+(m\sec\phi)}{L_+(1)}\right\} \frac{\tan\phi}{\sin\phi/2}\left(\frac{2m}{1+m}\right)^{1/2}.$$

SECTION 7

THE CASE WHEN $C_2 > C_1$

In the preceeding sections we have examined the case in which the initial disturbance passes into the medium in which the velocity of propagation is the greater. The reverse situation in which $m > 1$, is also of interest.

For the initial situation shown in figure 5, we find the solution in the hyperbolic region. This solution is shown in figure 8. In particular, we can see that the value of s_1 on the sonic circle in medium 1 is

$g(\mu_1 - \theta)$ for $0 > \theta > -\psi$

T for $-\psi > \theta > -\chi$, and

0 for $-\chi > \theta > -\pi$ where $\psi = \sec^{-1} m$ and $\chi = \sec^{-1} m \sec\phi$.

In medium 2 on the sonic circle the value of s_2 is

$1+R$ for $0 < \theta < \phi$ and

2 for $\phi < \theta < \pi$

The quantities T and R are the Fresnel coefficients defined previously in equation 14.

The triangle ABC, which results from the refraction of the disturbance in the elliptic region of medium 2 which travels at grazing incidence, is now in medium 1. Now the conditions to be imposed on $\partial\omega_2/\partial\zeta_2$ across the line AB is that

$$\partial\omega_2/\partial\zeta_2 = P(\zeta_2) \Big/ 1 - \frac{i}{k}\left(\frac{1-\zeta_2^2}{m^2\zeta_2^2-1}\right)^{1/2},$$

where $P(\zeta_2)$ is a function which takes real values on the real ζ_2-axis when $1/m < Re(\zeta_2) < 1$. To remove the unwanted branch points at the points $\zeta_2 = -1/m$, -1, we must introduce the factor $\exp + M(m\zeta_2)$, where

$$\pi M_+(\zeta) = \int_1^m \tan^{-1}\left\{\frac{(m^2-z^2)}{mk(z^2-1)}\right\}^{1/2} \Big/ \frac{dz}{z+\zeta} \qquad (30)$$

This function is derived exactly as in section 4 and its properties are very similar. We may now write down an expression for $\partial\omega_2/\partial\zeta_2$ which satisfies all the necessary conditions, and as before, we are only left with one constant to be determined. Thus

$$m^2\pi \frac{\partial\omega_2}{\partial\zeta_2} = \frac{B \exp M_+(m\zeta_2)}{m\left[1 - \frac{i}{k}\left(\frac{1-\zeta_2^2}{m^2\zeta_2^2-1}\right)^{1/2}\right]} \frac{\left(\frac{m\zeta_2-1}{m\zeta_2}\right)^{1/2}}{(1-\zeta_2^2)^{1/2}(\zeta_2-\sec\phi)(\zeta_2-1)} \tag{31}$$

It is necessary to make $\partial\omega_2/\partial\zeta_2$ imaginary on the real ζ_2-axis when $0 < \zeta_2 <$ 1/m to avoid introducting branch points which give an incorrect solution. On applying the continuity conditions across OA we find that

$$\pi \frac{\partial\omega_1}{\partial\zeta_1} = \frac{-iB \exp M_+(\zeta_1)\ (1-\zeta_1/\zeta_1)^{1/2}}{\left[1+\frac{1}{mk}\left(\frac{m^2-\zeta_1^2}{1-\zeta_1^2}\right)^{1/2}\right](1-\zeta_1^2)^{1/2}(\zeta_1-m\sec\phi)(\zeta_1-m)} \tag{32}$$

and this expression satisfies all the required conditions.

The constant B, which is found by the method of residues is given by the equation

$$B = \alpha m^2 \exp - M_+(m\sec\phi)\,(\sec\phi-1)\tan\phi\left[\frac{m\sec\phi+1}{m\sec\phi-1}\right]^{1/2} \tag{33}$$

This completes the description of the calculation of the velocity and condensation (or pressure) derivatives in the acoustic problem, when the angle of incidence, as shown in figure 5, lies in the range $0 < \phi < \pi/2$. The results are directly applicable to the electromagnetic problem in the magnetic mode, with a perfectly conducting half-plane provided that we take the constant k to define the ratio of the dielectric constants of the two media instead of the density ratio.

SECTION 8

THE PERFECTLY ABSORBENT HALF-PLANE

When we consider the problem in which s vanishes at the surface of the half-plane, there is only a slight change in the method of solution. To satisfy this boundary condition the reflected pulse must annul instead of reinforce the incident pulse. The initial situation must be changed accordingly, as in figure 10. For the solution in hyperbolic region in the various cases we may refer to the figures 6, 7, and 8, if we reduce the amplitude of the disturbance behind the front ER by 2.

Within the elliptic regions the vanishing of s on $\theta = \pm\pi$ means that $\partial s/\partial\lambda = 0$ on the line $\lambda < c$, $\theta = \pm\pi$. Hence $\partial\omega/\partial\zeta$ must be imaginary on the real axis for $-1 < \zeta_1 < 0$. All the conditions of the problem may be satisfied by the expressions

$$\pi \frac{\partial\omega_1}{\partial\zeta_1} = \frac{A \exp - L_+(\zeta_1) \left(\frac{1-\zeta_1}{\zeta_1}\right)^{1/2}}{\left[1+\frac{i}{mk}\left(\frac{\zeta_1^2-m^2}{1-\zeta_1^2}\right)^{1/2}\right](\zeta_1 - m\sec\phi)(\zeta_1-1)}, \tag{34}$$

and

$$m^2 k\pi \frac{\partial\omega_2}{\partial\zeta_2} = \frac{A \exp - L_+(m\zeta_2) \left(\frac{1-m\zeta_2}{m\zeta_2}\right)^{1/2}}{\left[1+\frac{1}{k}\left(\frac{1-\zeta_2^2}{1-m^2\zeta_2^2}\right)^{1/2}\right](\zeta_2 - \sec\phi)(m\zeta_2-1)}. \tag{35}$$

No solution which has an imaginary part on the interface between the two elliptic regions is correct, since it must then, according to the continuity conditions 19, contain unwanted branch points in one medium or the other.

When $m \sec\phi > 1$ the constant A must take the value

$$2m^2 \exp L_+(m\sec\phi)\,\tan\phi\left(\frac{m\sec\phi}{m\sec\phi+1}\right)^{1/2}.$$

When $m \sec\phi > 1$, total reflexion is expected, and when we use the method of residues to find the constant A and its relation to the jump across the front reflected on the interface, we find that this value for A is unchanged.

When $m > 1$ as in figure 8 the results in the elliptic regions are that

$$m^3 \pi \, \partial\omega_2/\partial f_2 = \frac{A \exp M_+(m f_2) \left(\frac{1-f_2}{f_2}\right)^{1/2}}{\left[1 - \frac{i}{k}\left(\frac{1-f_2^2}{m^2 f_2^2 - 1}\right)^{1/2}\right](f_2 - \sec\phi)(f_2 - 1)}, \tag{37}$$

and

$$\pi \, \partial\omega_1/\partial f_1 = \frac{A \exp M_+(f_1) \left(\frac{m - f_1}{f_1}\right)^{1/2}}{\left[1 + \frac{1}{mk}\left(\frac{m^2 - f_1^2}{1 - f_1^2}\right)^{1/2}\right](f_1 - m\sec\phi)(f_1 - m)},$$

Here A must have the value

$$2 m^3 \tan\phi \left[\frac{\sec\phi(\sec\phi - 1)}{m^2 \sec^2\phi - 1}\right]^{1/2} \exp - M_+(m \sec\phi) .$$

These results may also be used in the electromagnetic problem in the electric mode, with a perfectly conducting half-plane, if we take s to be the quantity $E_z/\mu c^2$ and if we replace the density ratio k by the ratio of the permeabilities μ of the two media.

CONCLUSION

By assuming dynamic similarity in the solution, we have found explicit formulae for the derivatives of the field components in a series of pulse diffraction problems. These results show clearly how the pulses are propagated and also how a refracted disturbance is set up even in the case of total reflexion. A most interesting result is that the dynamically similar solution for the problem of diffraction by a half-plane $r > o$, $\theta = \pm\pi$ has the same form of solution on the half-plane $\theta = 0$ within the smaller sonic circle whether the media in the two regions $0 > \theta > -\pi$ and $0 < \theta < \pi$ have distinct properties or not. This statement is the immediate consequence of the fact that $\partial\omega/\partial\xi$ is either real in the absorbent half plane problem or pure imaginary in the other case. The results in the homogeneous medium follow directly from equations 11 and 13. Thus for the reflecting half-plane the pressure is constant, and for the absorbing half-plane the normal velocity vanishes on a steadily expanding section of the half-plane, $\theta = 0$.

The method described may also be used to analyze the case when the incident pulse passes into a medium in which the velocity of propagation depends on the direction of motion of the disturbance.

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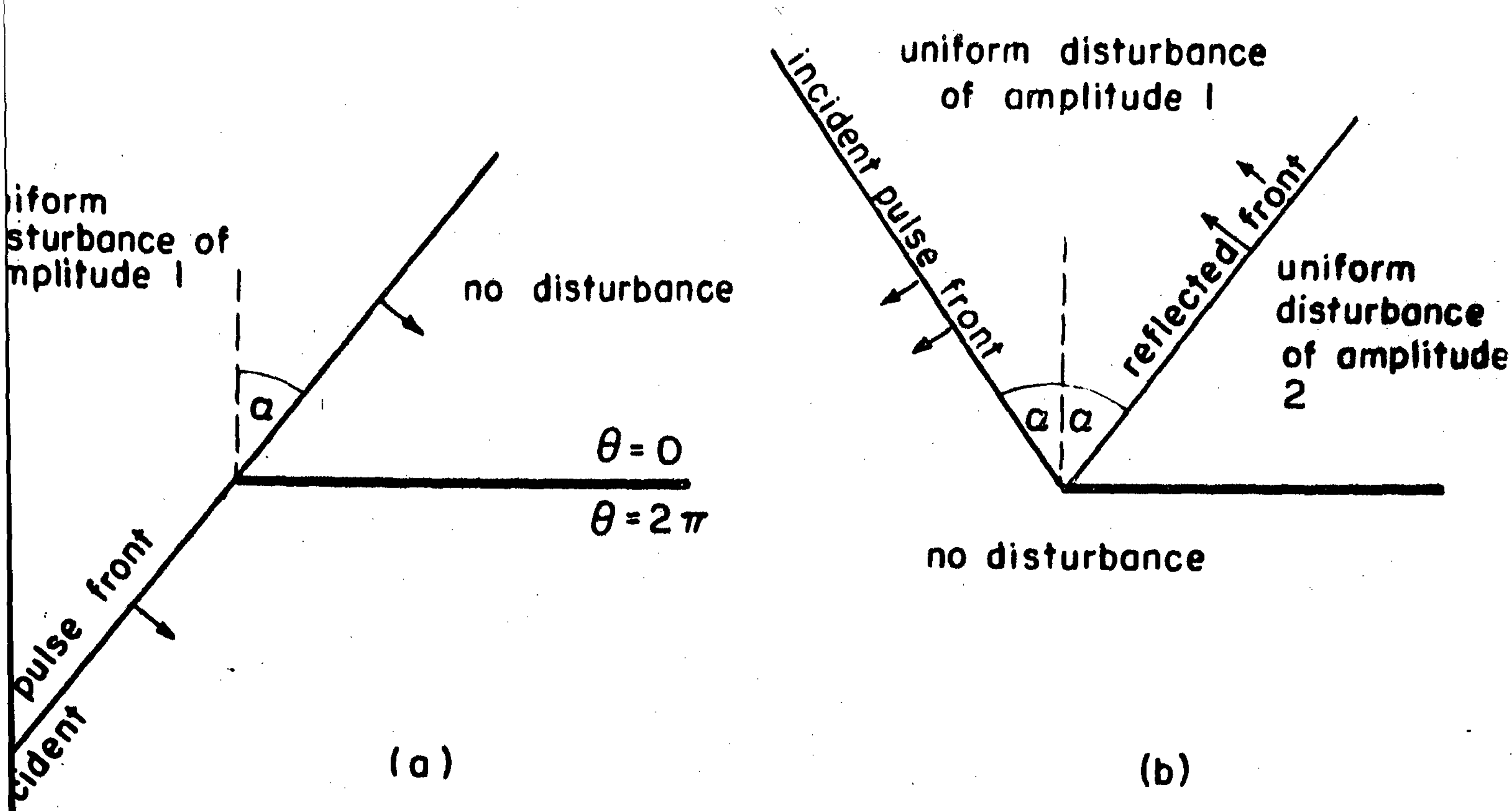

FIG. 1 INITIAL STATES IN THE DIFFRACTION OF A PULSE BY A REFLECTING HALF PLANE

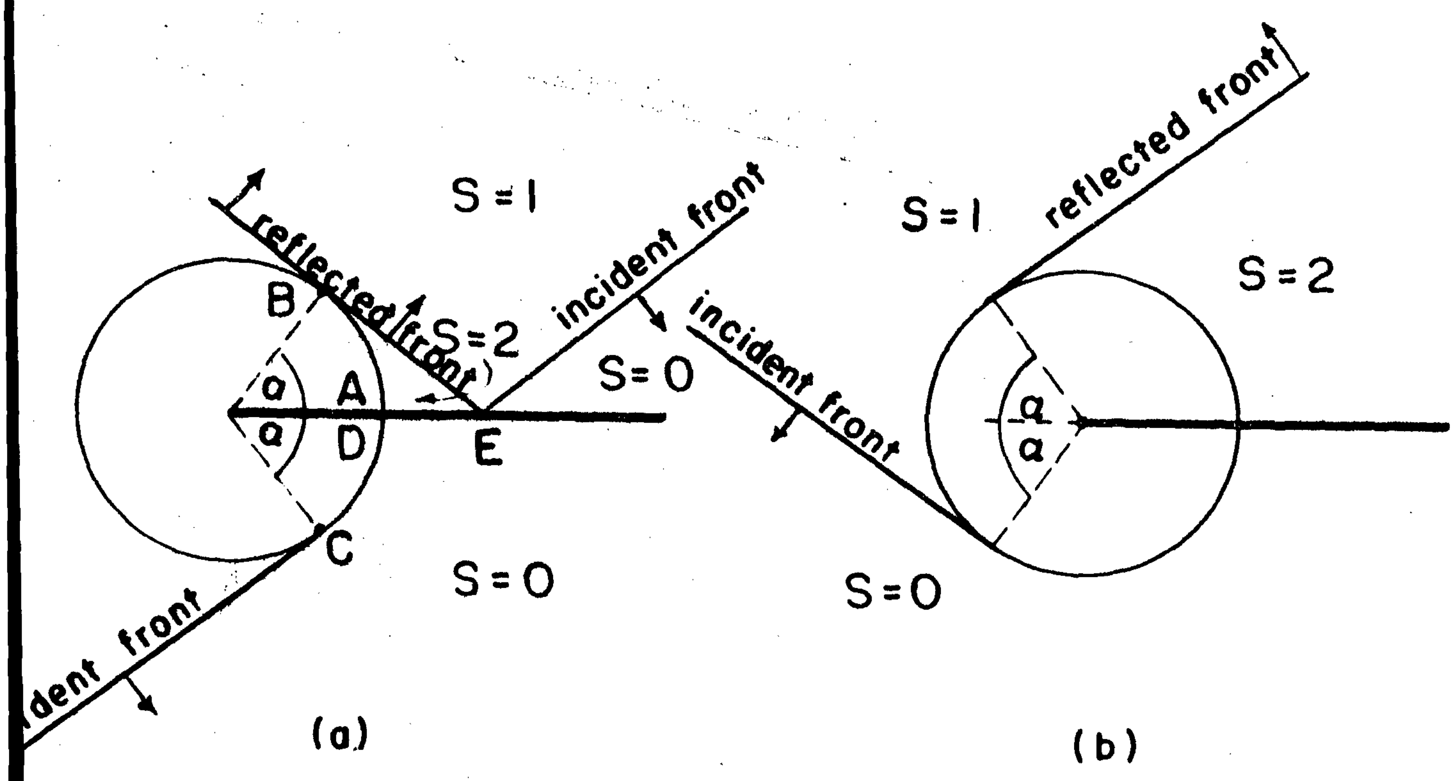

FIG. 2 THE SOLUTION IN THE HYPERBOLIC REGION OF THE (λ, θ) PLANE

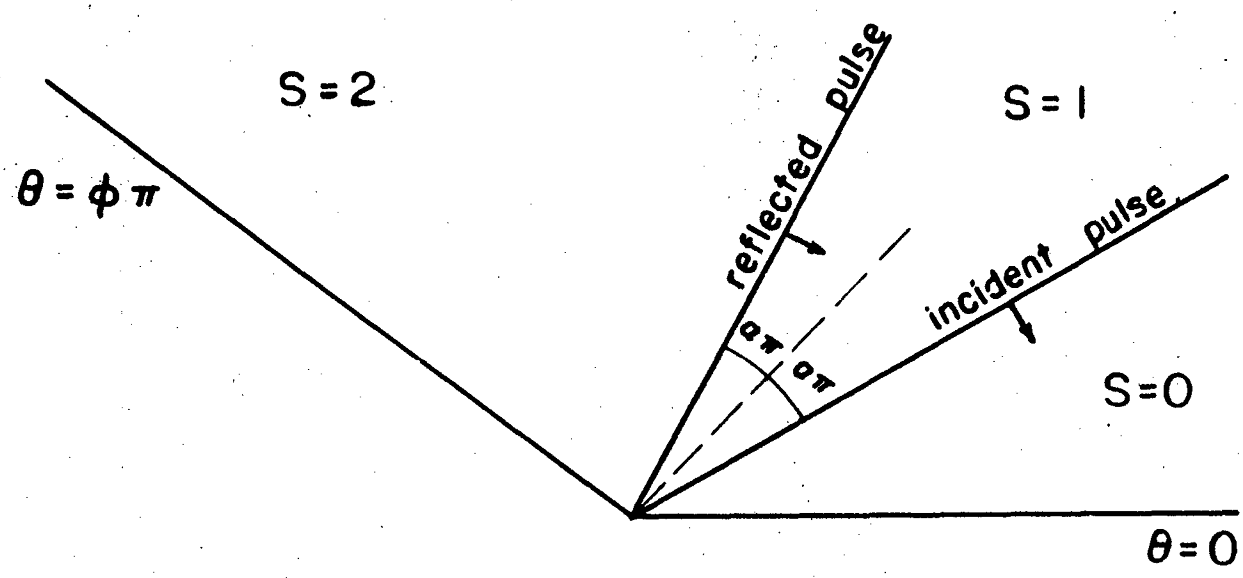

FIG. 3. THE INITIAL STATE IN THE WEDGE DIFFRACTION PROBLEM

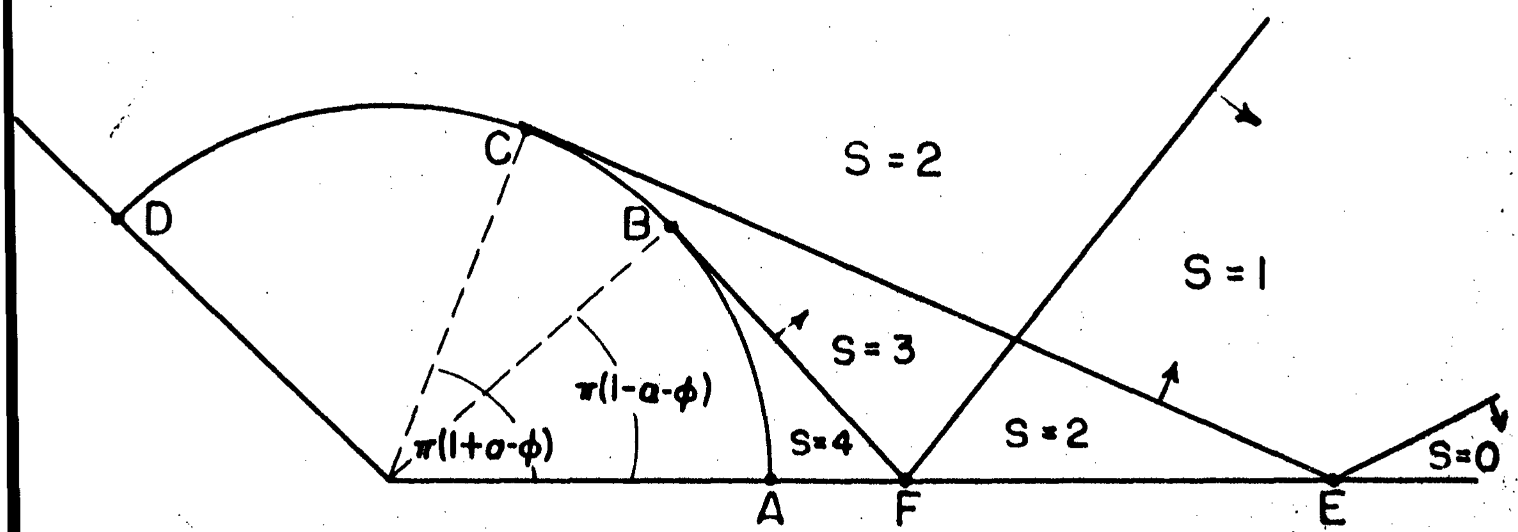

FIG. 4 THE SOLUTION IN THE HYPERBOLIC ORIGIN OF THE (λ, θ) PLANE

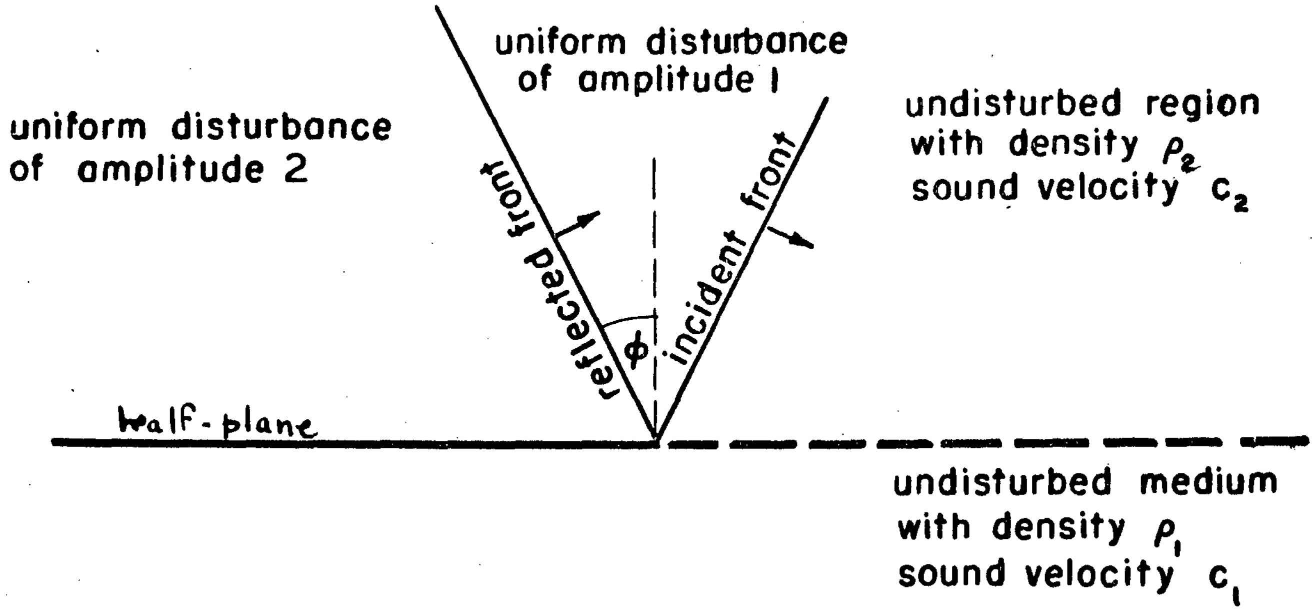

FIG. 5. THE INITIAL STATE IN THE DIFFRACTION AND REFRACTION OF A PULSE BY A PERFECTLY REFLECTING HALF-PLANE

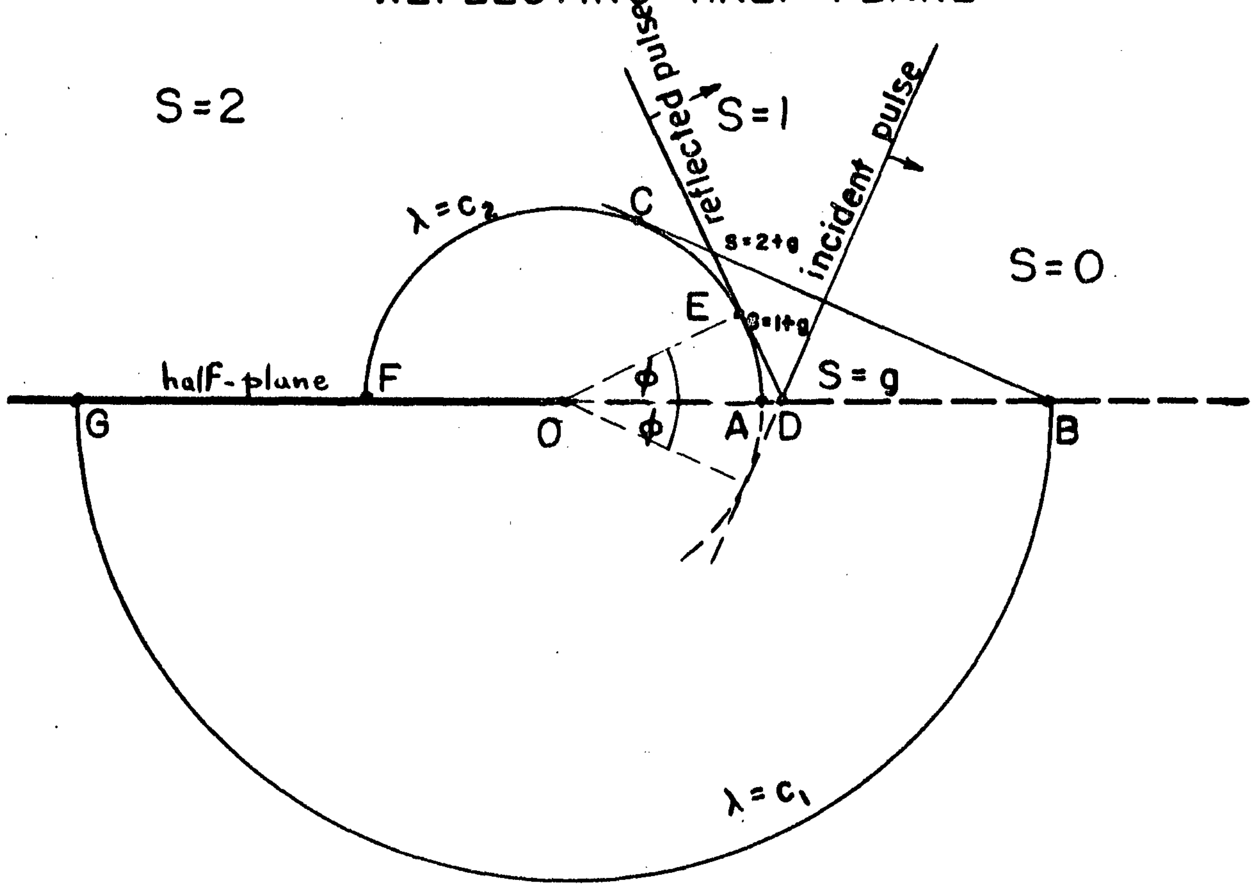

FIG. 6 THE SOLUTION IN THE HYPERBOLIC REGION OF THE (λ,θ) PLANE WITH $c_2 > c_1$ FOR SUB-CRITICAL INCIDENCE $c_2 \sec\phi < c_1$

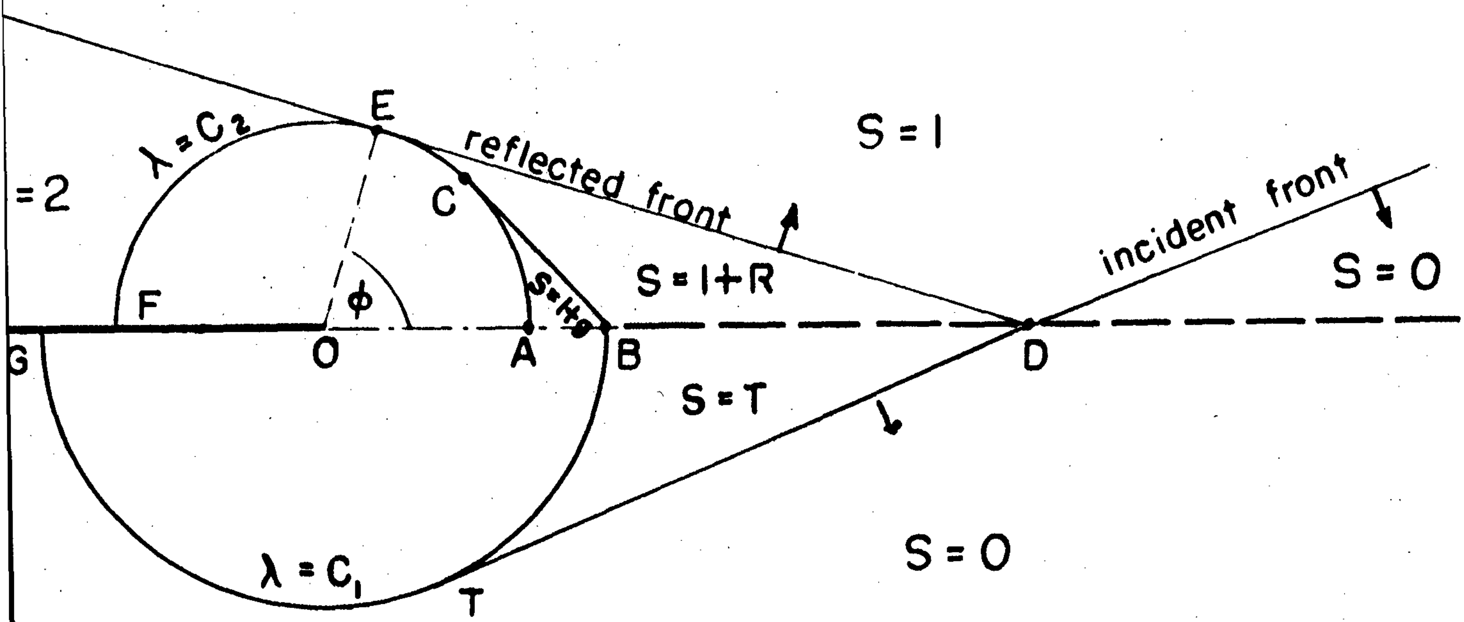

FIG. 7. THE SOLUTION IN THE REGION OF THE (λ, θ) PLANE WITH $c > c$ FOR SUPER-CRITICAL INCIDENCE $c_2 \cos\phi > c_1$

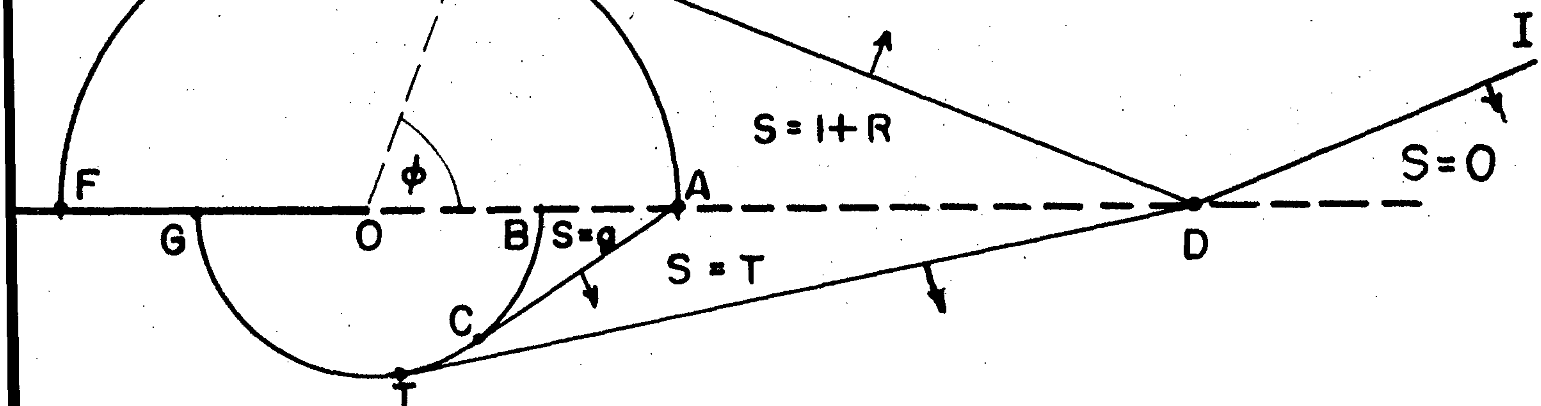

FIG. 8 THE SOLUTION IN THE HYPERBOLIC REGION OF THE (λ, θ) PLANE WITH $c_1 < c_2$

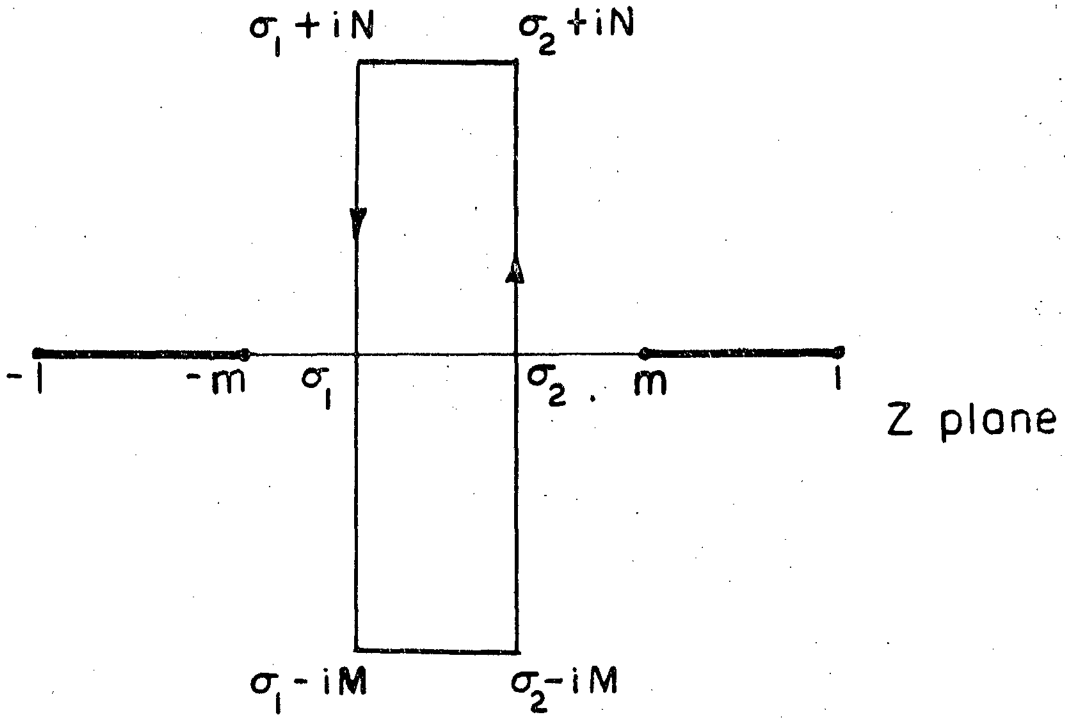

FIG. 9 THE CONTOUR C OF INTEGRATION IN THE COMPLEX Z-PLANE

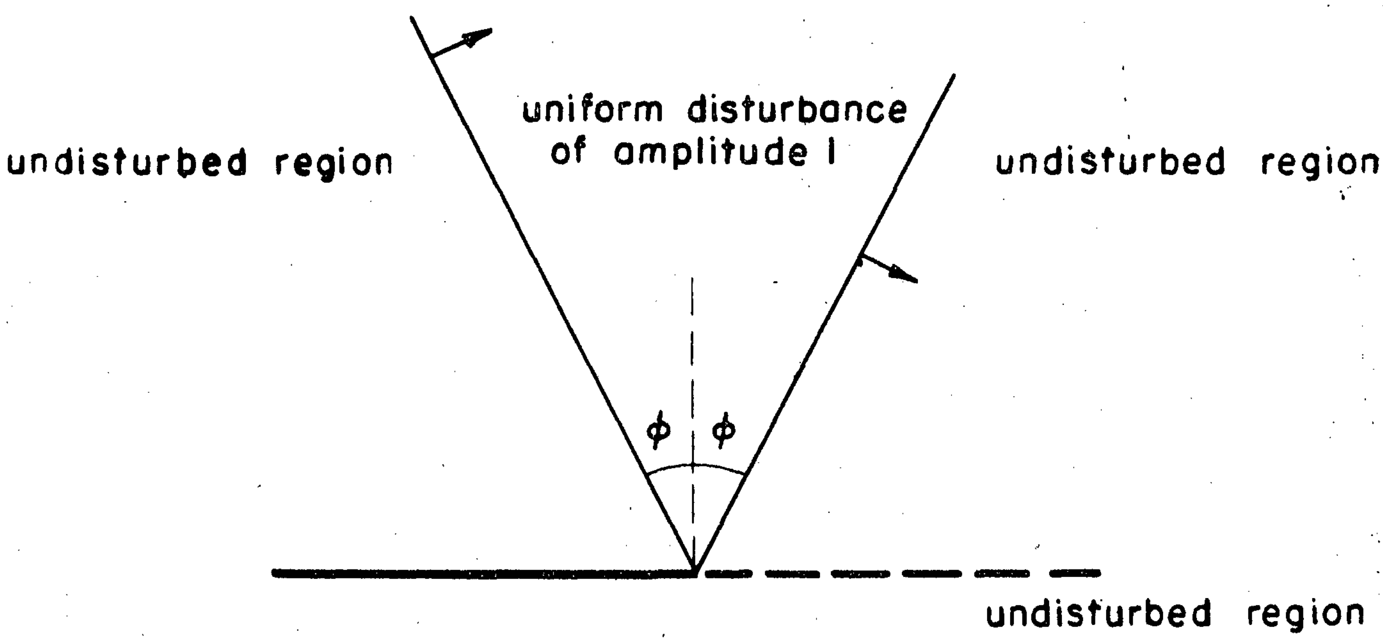

FIG. 10 THE INITIAL STATE FOR A PERFECTLY ABSORBING HALF-PLANE

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